Due by March 9, 23:59 pm.

Exercise 1 (10 points)

Demonstrate what happens when we insert the keys $5, 28, 19, 15, 20, 33, 12, 17, 10$ (in this order) into a hash table with collisions resolved by chaining. Let the hash table have 9 slots and let the hash function be $h(k) = k \bmod 9$.

Exercise 2 (15 points)

Assume we are storing a set of n keys into a hash table of size m. Show that if the keys come from a universe U with $|U| > nm$, then U has a subset of size n consisting of keys that all hash to the same slot, so that the worst-case searching time for hashing with chaining is $\Theta(n)$.

Exercise 3 (15 + 10 points)

a) Let $p, m \in \mathbb{N}$ with p prime and $p > m > 1$. Let $r \in \{0, 1, \ldots, p-1\}$ and

$$S = \big\{s \in \{0, 1, \ldots, p-1\} : s \neq r \ \wedge \ s \equiv r \pmod{m}\big\}.$$

Show that $|S| \leq \lceil \frac{p}{m} \rceil - 1$.

b) Let $a, b \in \mathbb{N}$. Show that

$$\left\lceil \frac{a}{b} \right\rceil \leq \frac{a}{b} + \frac{b-1}{b}.$$

Exercise 4 (50 points)

Let $\mathcal{H}$ be a finite collection of hash functions from a given universe U of keys to $\{0, 1, \ldots, m-1\}$. We call $\mathcal{H}$ universal if for all $k_1, k_2 \in U$ with $k_1 \neq k_2$

$$\Pr_h\left[h(k_1) = h(k_2)\right] \leq \frac{1}{m},$$

where the probability is over h chosen uniformly at random from $\mathcal{H}$.

Consider $U = \{0, 1\}^u$ for some $u \in \mathbb{N}$ and let $m = 2^b$ for some $b \in \mathbb{N}$. For $A \in \{0, 1\}^{b \times u}$ let h_A be

$$h_A : U \to \{0, 1, \ldots, m-1\}, \quad h_A(k) = \sum_{i=1}^{b} 2^{i-1} \left[\Big(\sum_{j=1}^{u} A_{ij} \cdot k_j \Big) \bmod 2 \right],$$

where $k = (k_1, k_2, \ldots, k_u)$. Show that $h_A \neq h_B$ for $A \neq B$ and that $\mathcal{H} = \{h_A : A \in \{0, 1\}^{b \times u}\}$ is universal.