Due by April 27, 23:59 pm.

Exercise 1 (20 points)

Let P be the transition matrix of a Markov chain. Prove that the ij-th entry $p_{ij}^{(n)}$ of P^n gives the probability that the Markov chain, when starting in state s_i, will be in state s_j after n steps, that is

$$p_{ij}^{(n)} = (P^n)_{ij} = \Pr[X_n = s_j \mid X_0 = s_i].$$

Exercise 2 (10 + 10 + 10 points)

Consider two Markov chains with the following transition matrices P_1 and P_2:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

a) Which of the Markov chains is regular?

b) For both Markov chains, provide all stationary distributions.

c) Does P_1^n or P_2^n converge as $n \to \infty$? If so, provide the limit.

Exercise 3 (50 points)

A Markov chain is called irreducible if all states are reachable from all other states. That is, if $P \in [0,1]^{r \times r}$ is the transition matrix, then for all $i, j \in \{1, \ldots, r\}$ there exists some $t \in \mathbb{N}$ such that $p_{ij}^{(t)} = (P^t)_{ij} > 0$.

Prove that a Markov chain is irreducible if and only if there is no permutation matrix $Q \in \{0,1\}^{r \times r}$ such that

$$Q^T P Q = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$$

for some non-trivial square matrices P_{11} and P_{22}.